Solution to Exercises – Chapter 5

1. In the case of both range-rate and state vector observables, the energy balance equations for the potential difference, without the specific force term and neglecting the time-integral and pole tide potential, is given by

$$\begin{aligned} V_{12}^{(\dot{\rho})}(t) = {} & \frac{1}{2}\dot{\rho}_{12}\left(\dot{\boldsymbol{x}}_1^e + \dot{\boldsymbol{x}}_2^e\right)^{\mathrm{T}} \boldsymbol{e}_{12} + \frac{1}{2}\left(\left|\boldsymbol{e}_n^{\mathrm{T}}\dot{\boldsymbol{x}}_2^e\right|^2 - \left|\boldsymbol{e}_n^{\mathrm{T}}\dot{\boldsymbol{x}}_1^e\right|^2\right) + \frac{1}{2}\left(\left|\boldsymbol{e}_r^{\mathrm{T}}\dot{\boldsymbol{x}}_2^e\right|^2 - \left|\boldsymbol{e}_r^{\mathrm{T}}\dot{\boldsymbol{x}}_1^e\right|^2\right) \\ & - \frac{\omega_E^2}{2}\left(\left|\boldsymbol{e}_3^e \times \boldsymbol{x}_2^e\right|^2 - \left|\boldsymbol{e}_3^e \times \boldsymbol{x}_1^e\right|^2\right) - E_{12}^{(0)} \end{aligned} \tag{1}$$

where ω_E is Earth's nominal rotation rate, $E_{12}^{(0)}$ is a constant, and along-track, cross-track, and radial unit vectors are defined by

$$\boldsymbol{e}_{12} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_1}{\left|\boldsymbol{x}_2 - \boldsymbol{x}_1\right|}, \quad \boldsymbol{e}_n = \frac{\boldsymbol{x}_1 \times \boldsymbol{x}_2}{\left|\boldsymbol{x}_1 \times \boldsymbol{x}_2\right|}, \quad \boldsymbol{e}_r = \boldsymbol{e}_{12} \times \boldsymbol{e}_n . \tag{2}$$

Without the range-rate observable, there is

$$V_{12} = \frac{1}{2}\left(\left|\dot{\boldsymbol{x}}_2^e\right|^2 - \left|\dot{\boldsymbol{x}}_1^e\right|^2\right) - \frac{\omega_E^2}{2}\left(\left|\boldsymbol{e}_3^e \times \boldsymbol{x}_2^e\right|^2 - \left|\boldsymbol{e}_3^e \times \boldsymbol{x}_1^e\right|^2\right). \tag{3}$$

2. Any high level computer programming language may be used to calculate the right-hand sides of (1) and (3).

3. A plot of the along-track, cross-track, radial, and rotation potential terms is shown in Figure 1. The cross-track and rotation potential differences are almost the same because in the e-frame the cross-track velocity essentially is the velocity of Earth's rotation (since the orbits are almost polar). And, the cross-track velocity is responsible for the centrifugal potential.

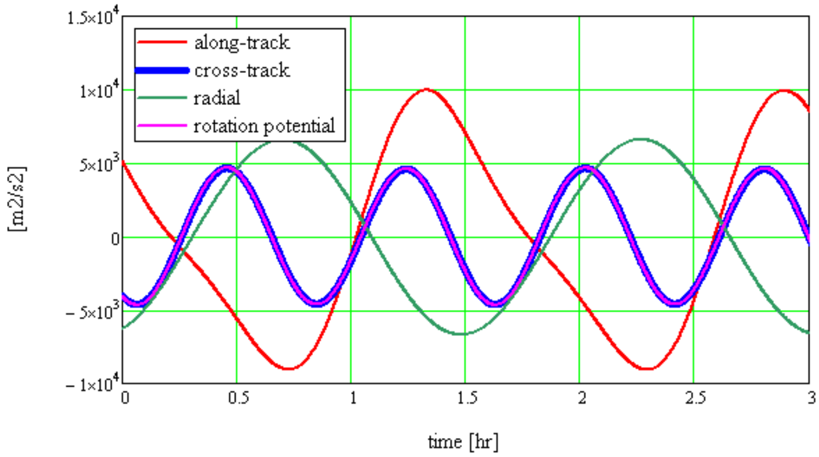

Figure 1: The individual terms of the right side of (1).

4. Comparisons of the potential differences from the energy balance equations (1) and (3) and EGM2008 are shown in Figure 2.

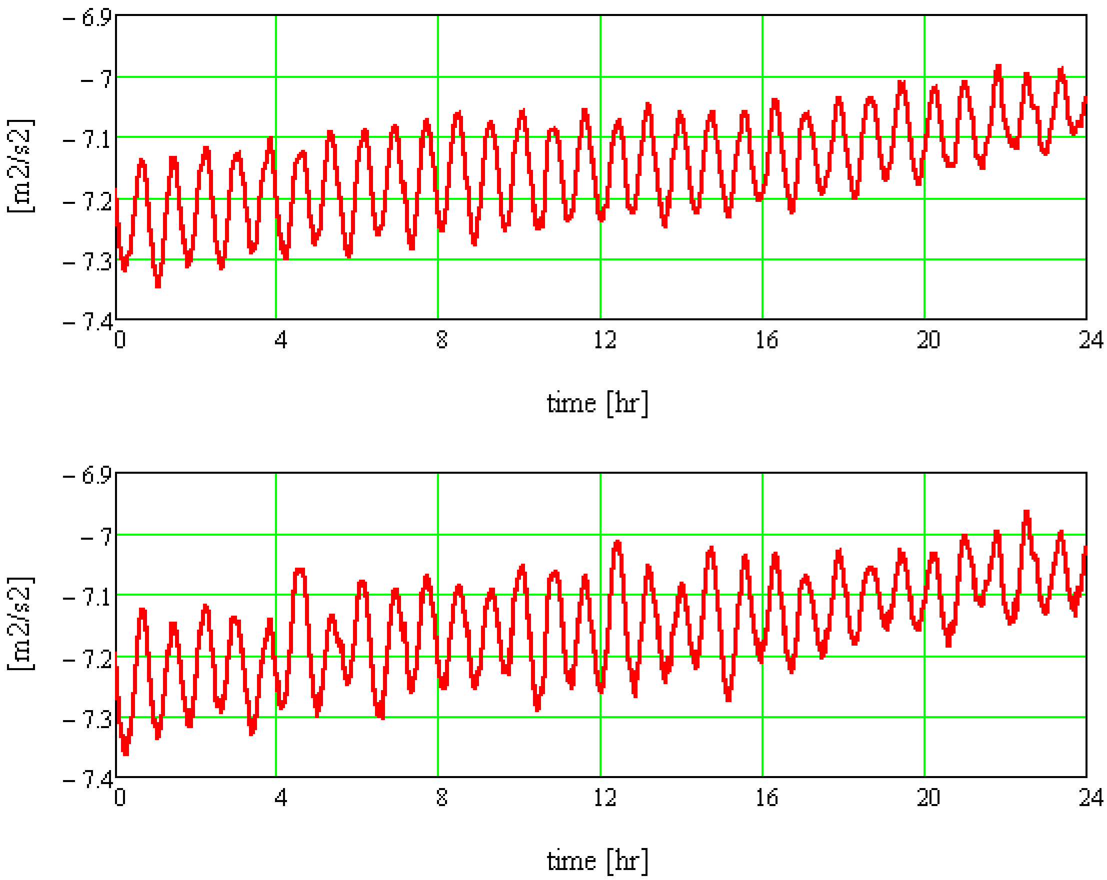

Figure 2: Differences $V_{12} - V_{12}^{(EGM\,2008)}$ using (3) (top) and $V_{12}^{(\dot{\rho})} - V_{12}^{(EGM\,2008)}$ (1) (bottom).

The mean and standard deviation of the differences are,

$$\mu = -7.151 \text{ m}^2/\text{s}^2, \quad \sigma = 0.077 \text{ m}^2/\text{s}^2, \tag{4}$$

$$\mu^{(\dot{\rho})} = -7.152 \text{ m}^2/\text{s}^2, \quad \sigma^{(\dot{\rho})} = 0.079 \text{ m}^2/\text{s}^2. \tag{5}$$

We see that there is virtually no difference in terms of the statistics. The mean represents the constant energy difference, $E_{12}^{(0)}$. The oscillations are in part due to the neglected tidal potential difference; however, see also part 5, below.

5. The average formal standard deviations of the positions and velocities are

$$\sigma_x \simeq 3.6\times10^{-3} \text{ m}, \quad \sigma_{\dot{x}} \simeq 3.45\times10^{-6} \text{ m/s}. \tag{6}$$

Assuming that the range-rate accuracy is $\sigma_{\dot{\rho}_{12}} = 0.1\times10^{-6}$ m/s, then the simple error analysis (Figure 5.13) shows that the contributions to the potential difference accuracy are approximately $0.045 \text{ m}^2/\text{s}^2$, $0.0025 \text{ m}^2/\text{s}^2$, and $0.0005 \text{ m}^2/\text{s}^2$. Thus the total estimated potential difference accuracy per point is about $0.045 \text{ m}^2/\text{s}^2$, due mostly to position error. This is of the same order of magnitude as the standard deviation of the "error" in (5), where the latter also includes the effect of the neglect of the temporally varying potentials (tides, etc.). However, this also suggests that the errors seen in Figure 2 are largely due to position errors in the rotation potential term.

6. The median-smoothed periodograms of the differences, $V_{12} - V_{12}^{(\text{EGM2008})}$, and $V_{12}^{(\dot{\rho})} - V_{12}^{(\text{EGM2008})}$ are shown in Figure 3. The resonant peak is at roughly 2 cycles per revolution which is likely due to omitted tides. The potential differences that include the range-rate observable have extra power relative to the EGM2008($n_{\max} = 180$) model at frequencies above $f \simeq 3\times10^{-2}$ Hz, which, given the approximate speed of the satellites, $v = 7628$ m/s, corresponds to spatial frequencies greater than $\eta \simeq 3.93\times10^{-6}$ cy/m, or half-wavelengths shorter than $s = 127$ km, or harmonic degree larger than $n = R\pi/s = 6844000\pi/127000 = 170$, which is just at the limit of the EGM2008 "truth". Therefore, the added power is likely due to noise in the KB range-rate measurement, rather than omitted "truth" or neglected effects. The same figure is obtained if the "truth" field is EGM2008($n_{\max} = 720$). The fact that the KBR seems not to add anything to the determination of potential differences is that the orbits are reduced dynamic orbits that are determined using a pre-defined global gravity model (like EGM2008). The result would be different if the orbits were strictly geometric (but then one would have to determine velocities from the positions).

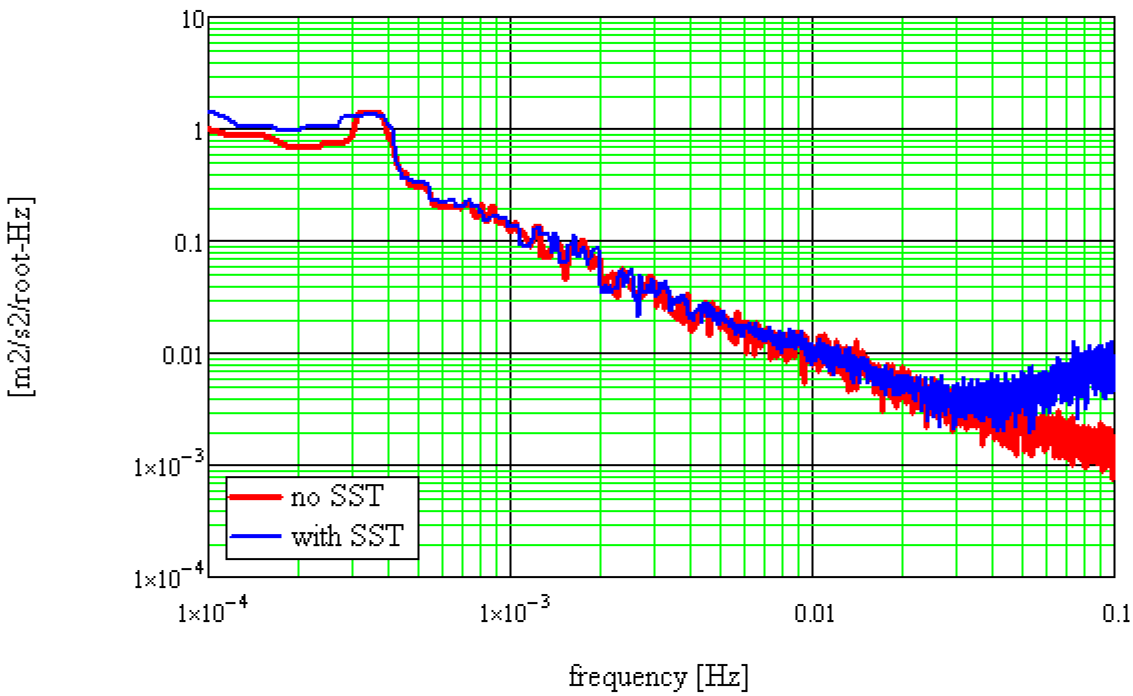

Figure 3. Median-smoothed psd's of the differences, $V_{12} - V_{12}^{(\mathrm{EGM2008})}$ (red), and $V_{12}^{(\dot{\rho})} - V_{12}^{(\mathrm{EGM2008})}$ (blue).